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2010 J. Phys. A: Math. Theor. 43 025305

(http://iopscience.iop.org/1751-8121/43/2/025305)

J. Phys. A: Math. Theor. **43** (2010) 025305 (11pp) doi:10.1088/1751-8113/43/2/025305

Output entanglement from $SU(1,1)$ coherent states under nonlinear dissipation in the dispersive limit

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Received 6 July 2009, in final form 28 October 2009
Published 10 December 2009
Online at stacks.iop.org/JPhysA/43/025305

Abstract
An analytical solution is found for the master equation of a system described by a nonlinear Jaynes–Cummings model, in the presence of nonlinear quantum dissipation at zero temperature in the large detuning approximation. We study the influence of nonlinear quantum dissipation on the output entanglement dynamics of the atom–field system, considering the field to be initially in $SU(1,1)$ coherent states. It is found that in the presence of the nonlinear quantum dissipation, the amplitude of the output entanglement between the field and the atom decreases with time, however the entanglement between the field–atom system and the environment increases with time.

PACS number: 03.65.Ud

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement is the distinguishing key feature of quantum mechanics setting it apart from classical physics. A quantum state of a composite system, consisting of two or more subsystems, is entangled if it cannot be factorized into a direct product of the states of the subsystems. Entangled states are useful in quantum information processing such as quantum teleportation [1], quantum key distribution [2] and superdense coding [3]. Studying quantum information theory using entangled coherent states has recently received much attention [4–8]. The coherent states of the $SU(2)$ and the $SU(1,1)$ algebras were studied [9] and generalized coherent states for $SU(n)$ were considered [10]. Also matrix elements of generalized coherent operators based on Lie algebras $SU(1,1)$ and $SU(2)$ were determined [11]. These states are interesting by themselves, moreover, they have very interesting applications. However, there are two commonly considered coherent states for $SU(1,1)$. One $SU(1,1)$ coherent state is the analogue of the harmonic oscillator coherent state achieved by displacing the vacuum state. The analogous $SU(1,1)$ coherent state is obtained via an $SU(1,1)$ transformation of lowest weight state. This $SU(1,1)$ coherent state is a member of Perelomov's category of a generalized coherent state, and we refer to this state as a Perelomov $SU(1,1)$ coherent state [12, 13]. The second $SU(1,1)$ coherent state, introduced by Barut and Girardello [14], is the analogue of the harmonic oscillator coherent state being an eigenstate of the annihilation operator. The purpose of the present work is to study entanglement induced by $SU(1,1)$ coherent states with nonlinear dissipation interacting with a two-level atom in the dispersive limit.

One of the generalizations of the Jaynes–Cummings (JC) model is the intensity-dependent Jaynes–Cummings (IDJC) model [15]. Because of the commensurability of the Rabi frequencies, which arises from such a coupling, this model presents absolutely periodic revivals, contrary to what happens in the ordinary JC model. Moreover, the state vector representing the evolution of the system is periodic itself. This means that there will be periodic evolution for any expectation value. What has not been acknowledged is that this behaviour leads to such an enhancement of certain effects that would otherwise be difficult to note within the realm of the original JC model. Because of this enhancement, it is possible to have the generation of well-defined Schrödinger cat-like states during the evolution of the field in the IDJC model, as has been already discussed [16]. A nonlinear IDJC model has received much attention in view of its connection with quantum algebras [17]. The quantum algebras, introduced as a mathematical description of deformed Lie algebras, have given the possibility of generalizing the notion of creation and annihilation operators of the usual quantum oscillator and to introduce the deformed oscillator.

Over the last two decades much attention has focused on the properties of the dissipative variants of the usual (non-deformed) JC model. The dissipative effects caused by the energy exchange between the system and the environment have been studied both analytically [18] and numerically [19]. In the last few years the JC model with phase damping (which occurs when there is no energy exchange between the system and the environment) has also been studied intensively and applied to decoherence and entanglement [20–22]. Furthermore, with the experimental realization of the two-photon micromaser [23] the dissipative two-photon JC model has attracted a great deal of attention. All of the above-cited studies have shown that the dissipation effects markedly change the dynamical behaviour of the atom–field system. We study the system in which an atom is coupled with the field via the interaction governed by a nonlinear JC model by making use of the dynamical algebraical method [24] and find the exact solution of the master equation for the system with phase decoherence.

In this paper, we investigate the entanglement properties of the $SU(1,1)$-related coherent field interacting with an atom in the dispersive approximation with nonlinear dissipation, and examine the influence of the initial states on entanglement.

2. The physical model and its analytic solution with Lie algebra

First, we consider a single-mode field interacting with an effective two-level atom with an intensity-dependent coupling without considering the influence of the dissipation. The Hamiltonian of the system under the rotating wave approximation is

$$\hat{H} = \tfrac{1}{2}\hbar\omega_{\circ}\sigma_z + \hbar\omega\hat{a}^{\dagger}\hat{a} + g\hbar[\hat{a}\sqrt{\hat{a}^{\dagger}\hat{a}}|e\rangle\langle g| + \sqrt{\hat{a}^{\dagger}\hat{a}}\hat{a}^{\dagger}|g\rangle\langle e|], \tag{1}$$

where ω is the frequency of the cavity field; $\omega_{\circ}$ is the transition frequency between the excited and ground states of the atom; $\hat{a}$ and $\hat{a}^{\dagger}$ are the annihilation and the creation operators of the cavity field respectively; g is the atom–field coupling constant; $\hat{\sigma}_z$ is the population inversion operator; and $\hat{\sigma}_+, \hat{\sigma}_-$ are the Pauli raising and lowering operators respectively, with the detuning parameter $\Delta = \omega_{\circ} - \omega$, which measures how far from resonance the two subsystems are. Such a generalization is of considerable interest because of its relevance to the study of the nonlinear coupling between a single atom and the radiation field whereas the atom makes single-photon transitions [15, 16]. It is also worth mentioning that generalized JC models have become the subject of intense attention [25–28]. These considerations support the theoretical interest in the IDJC model since this kind of interaction effectively means that the coupling constant is proportional to the intensity of the cavity field, which represents a very simple case of a nonlinear interaction corresponding to a more realistic physical situation. Moreover, the IDJC model is of considerable interest because of its relevance to the study of the intensity-dependent interaction between a single atom and the radiation field in quantum optics [15] as well as the study of the quantized motion of a single ion in an anharmonic-oscillator potential trap [29]. Physically, this model may be realized as if the cavity contains two different species of Rydberg atoms, one of which behaves like a two-level atom and the other behaves like an anharmonic oscillator in the single-mode field of frequency ω [30].

In the large detuning approximation, we have $\frac{|\Delta|}{g} \geqslant (n+1)$ for any 'relevant' photon number n, which means that the atom is in a cavity whose single-frequency photons are far from resonance. We can obtain an effective interaction Hamiltonian in the dispersive limit in the following form [31–35]:

$$\hat{H}_{\text{eff}} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{\omega_\circ}{2}\sigma_z + \lambda[(\hat{a}\hat{a}^\dagger)^2|e\rangle\langle e| - (\hat{a}^\dagger\hat{a})^2|g\rangle\langle g|], \tag{2}$$

where $\lambda = \frac{g^2}{\Delta}$. This Hamiltonian is nonlinear in the atomic space and, as was shown earlier [36]. It leads to a number of collective effects such as the atomic Schrödinger cat generation and atomic squeezing, and also it describes the dispersive evolution of the field. Recently, the dispersive limit of the JC model was studied [37, 38] in order to find relations between the entanglement of the atomic and field degrees of freedom and the decoherence caused by coupling the field mode to a zero-temperature reservoir (with linear quantum dissipation).

The $SU(1,1)$ Lie algebra for the single-mode field may be realized by introducing the operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ defined by the relations

$$\hat{K}_+ = \sqrt{\hat{a}^\dagger\hat{a}}\hat{a}^\dagger, \qquad \hat{K}_- = \hat{a}\sqrt{\hat{a}^\dagger\hat{a}}, \qquad \hat{K}_0 = \hat{a}^\dagger\hat{a} + \tfrac{1}{2}. \tag{3}$$

These three operators form a closed-$SU(1,1)$ Lie algebra, which is defined by the commutation relations [12]:

$$[\hat{K}_-, \hat{K}_+] = 2\hat{K}_0 \qquad [\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm. \tag{4}$$

In terms of the $SU(1,1)$ generators, we may rewrite the Hamiltonian in (2) as

$$\hat{H}_{\text{eff}} = \omega\left(\hat{K}_0 - \frac{1}{2}\right) + \frac{\omega_o}{2}\sigma_z + \lambda(\hat{K}_-\hat{K}_+|e\rangle\langle e| - \hat{K}_+\hat{K}_-|g\rangle\langle g|) = \hat{H}_0 + \hat{H}'_{\text{eff}}, \tag{5}$$

where $\hat{H}_0$ describes the free subsystems and $\hat{H}'_{\text{eff}}$ the interaction between them.

2.1. The master equation and its analytic solution

The damping is important, however, because realistic cavities have finite Q and measuring processes will cause photons to leak. However, a quantum system used in quantum information processing inevitably interacts with the surrounding environments (or the thermal reservoirs), which take the pure state of the quantum system into a mixed state. Thus, analysing the entanglement decay induced by the unavoidable interaction of the interested systems with the environment is an important subject. Recently, a master equation for the harmonic oscillator in the presence of nonlinear quantum dissipation has been derived [39], for the case of an interaction of the oscillator with its environment. If the bath is at zero temperature, the master equation for nonlinear quantum dissipation in the interaction picture has the following form under the Born–Markov approximation [40]:

$$\frac{\partial \rho(t)}{\partial t} = \gamma(2\hat{a} f(\hat{a}^\dagger\hat{a})\rho f(\hat{a}^\dagger\hat{a})\hat{a}^\dagger - \hat{a}^\dagger\hat{a} f^2(\hat{a}^\dagger\hat{a})\rho - \rho\hat{a}^\dagger\hat{a} f^2(\hat{a}^\dagger\hat{a})). \tag{6}$$

If we take $f(\hat{a}^\dagger\hat{a}) = \sqrt{\hat{a}^\dagger\hat{a}}$, the evolution of the compound atom–field system in a dispersive nonlinear JC model and in the presence of the nonlinear dissipation at zero temperature can be written in the interaction picture with the operators K_+, K_- and K_0 as

$$\begin{aligned}\frac{\partial \hat{\rho}(t)}{\partial t} &= -\mathrm{i}[\hat{H}'_{\text{eff}}, \rho] + \gamma(2\hat{K}_-\rho\hat{K}_+ - \hat{K}_+\hat{K}_-\rho - \rho\hat{K}_+\hat{K}_-)\\ &= -\mathrm{i}\lambda(\hat{K}_-\hat{K}_+|e\rangle\langle e|\rho - \hat{K}_+\hat{K}_-|g\rangle\langle g|\rho - \rho\hat{K}_-\hat{K}_+|e\rangle\langle e| + \rho\hat{K}_+\hat{K}_-|g\rangle\langle g|)\\ &\quad + \gamma(2\hat{K}_-\rho\hat{K}_+ - \hat{K}_+\hat{K}_-\rho - \rho\hat{K}_+\hat{K}_-).\end{aligned} \tag{7}$$

The master equation can be solved by applying the superoperator method proposed in [33]. By applying the dynamical symmetry method proposed in [41], the solution of the master equation

$$\frac{\partial \hat{\varrho}}{\partial t} = (\hat{L} + \hat{J})\hat{\varrho}, \tag{8}$$

where

$$[\hat{L}, \hat{J}]\hat{\varrho} = \hat{N}\hat{J}\varrho, \qquad [\hat{N}, \hat{L}] = 0 \tag{9}$$

can be given in the form

$$\hat{\varrho}(t) = \mathrm{e}^{\hat{L}t}\,\mathrm{e}^{(\frac{1-\mathrm{e}^{-\hat{N}t}}{\hat{N}})\hat{J}}\hat{\varrho}(0), \tag{10}$$

where $\hat{L}$, $\hat{J}$ and $\hat{N}$ are the superoperators which satisfy the commutations relations (9). Therefore, the solution of the master equation (7) for any initial state $\rho(0)$ is given by

$$\hat{\rho}(t) = \begin{pmatrix} \mathrm{e}^{\hat{L}_{ee}t}\,\mathrm{e}^{(\frac{1-\mathrm{e}^{-\hat{N}t}}{\hat{N}})\hat{J}}\hat{\rho}_{ee}(0) & \mathrm{e}^{\hat{L}_{eg}t}\,\mathrm{e}^{(\frac{1-\mathrm{e}^{-\hat{R}t}}{\hat{R}})\hat{J}}\hat{\rho}_{eg}(0)\\ \mathrm{e}^{\hat{L}_{ge}t}\,\mathrm{e}^{(\frac{1-\mathrm{e}^{-\hat{R}^*t}}{\hat{R}^*})\hat{J}}\hat{\rho}_{ge}(0) & \mathrm{e}^{\hat{L}_{gg}t}\,\mathrm{e}^{(\frac{1-\mathrm{e}^{-\hat{N}^*t}}{\hat{N}^*})\hat{J}}\hat{\rho}_{gg}(0)\end{pmatrix}, \tag{11}$$

where

$$\begin{aligned}&\hat{L}_{ee}\hat{\rho} = -\epsilon\hat{K}_+\hat{K}_-\hat{\rho} - \epsilon^*\hat{\rho}\hat{K}_+\hat{K}_- - \mathrm{i}2\lambda(\hat{K}_0\hat{\rho} - \hat{\rho}\hat{K}_0),\\ &\hat{L}_{gg}\hat{\rho} = -\epsilon^*\hat{K}_+\hat{K}_-\hat{\rho} - \epsilon\hat{\rho}\hat{K}_+\hat{K}_-,\\ &\hat{L}_{eg}\hat{\rho} = -\epsilon\hat{K}_+\hat{K}_-\hat{\rho} - \epsilon\hat{\rho}\hat{K}_+\hat{K}_- - i2\lambda\hat{K}_o\hat{\rho},\\ &\hat{L}_{ge}\hat{\rho} = -\epsilon^*\hat{K}_+\hat{K}_-\hat{\rho} - \epsilon^*\hat{\rho}\hat{K}_+\hat{K}_- + \mathrm{i}2\lambda\hat{\rho}\hat{K}_o,\\ &\hat{J}\hat{\rho} = 2\gamma\hat{K}_-\hat{\rho}\hat{K}_+, \qquad \hat{N}\hat{\rho} = \Gamma\hat{\rho} + \hat{\rho}\Gamma^\dagger,\\ &\hat{R}\hat{\rho} = (\hat{\Gamma} + \mathrm{i}2\lambda)\hat{\rho} + \hat{\rho}\Gamma,\\ &\epsilon = (\gamma + \mathrm{i}\lambda), \qquad \hat{\Gamma} = 2\epsilon\hat{K}_o.\end{aligned} \tag{12}$$

It is easy to show that the superoperators of (12) obey the commutation relations

$$\begin{aligned}&[\hat{L}_{ee}, \hat{J}]\hat{\rho} = \hat{N}\hat{J}\hat{\rho}, \qquad [\hat{L}_{gg}, \hat{J}]\hat{\rho} = \hat{N}^*\hat{J}\hat{\rho},\\ &[\hat{L}_{eg}, \hat{J}]\hat{\rho} = \hat{R}\hat{J}\hat{\rho}, \qquad [\hat{L}_{ge}, \hat{J}]\hat{\rho} = \hat{R}^*\hat{J}\hat{\rho}.\end{aligned} \tag{13}$$

Also the following commutators apply:

$$[\hat{L}_{ee}, \hat{N}] = 0, \qquad [\hat{L}_{gg}, \hat{N}^*] = 0, \qquad [\hat{L}_{eg}, \hat{R}] = 0, \qquad [\hat{L}_{ge}, \hat{R}^*] = 0. \tag{14}$$

We apply the time-dependent analytical solution for the matrix elements for any initial cavity field and use it to calculate the time evolution properties of coherence and entanglement in the IDJC model in the dispersive approximation in the next section.

3. Dynamical properties of the entanglement

In order to gain insight into the processes of entanglement, dissipation and the relation between them, we evaluate the time evolution of any initial state for the atom and field. But here, the field is initially in a nonlinear coherent state, which can be classified as an algebraic generalization of the conventional coherent state. This state is defined as the right eigenstate of the annihilation operator K_-. Actually, the $SU(1,1)$ group coherent states have been known for many years under other names, such as the phase state or its generalization [42]. The physical meaning of $SU(1,1)$ coherent states has been elucidated in [43, 44], where it has been shown that such states may appear as stationary states of the centre-of-mass motion of a trapped ion [43], or may be related to some nonlinear processes (such as a hypothetical 'frequency blue shift' in high-intensity photon beams [44]). Furthermore, it has been shown that $SU(1,1)$ coherent states exhibit various non-classical features such as quadrature squeezing, number-phase squeezing and sub-Poissonian photon statistics. So, we evaluate the time evolution of the following initial state:

$$|\psi_{AF}\rangle = (\kappa_e|e\rangle + \kappa_g|g\rangle)|\alpha, k\rangle, \tag{15}$$

where, as is usual in experiments [45], the atom enters the cavity in a coherent superposition and finds there an $SU(1,1)$ coherent state $|\alpha, k\rangle$. Where $|\alpha, k\rangle$ is the $SU(1,1)$ coherent state. There are two distinct $SU(1,1)$ coherent states to consider.

3.1. Perelomov $SU(1,1)$ coherent states

The Perelomov coherent state of the $SU(1,1)$ algebra is the result of the displacement operator acting on the vacuum state, and is defined as [12, 13]

$$|\alpha, k\rangle_P = (1-|\alpha|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)n!}}\, \alpha^n |n, k\rangle, \tag{16}$$

where $\Gamma(\cdot)$ is the gamma function and k is the Bargmann number; it comes from the Casimir operator defined as $\hat{C}^2 = \hat{K}_0^2 - \frac{1}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+) = k(k-1)\hat{I}$. In this paper we set $k = \frac{1}{2}$ and $\alpha = |\alpha|\,\mathrm{e}^{\mathrm{i}\phi}$ $(|\alpha| < 1)$; in this case $\left|\alpha, \frac{1}{2}\right\rangle_P$ describes a pure thermal state.

3.2. Barut–Girardello $SU(1,1)$ coherent states

There is another coherent state of the $SU(1,1)$ algebra known as the Barut–Girardello coherent state [14]. It is defined as [46] the eigenstate of the lowering operator K_-:

$$K_-|\alpha, k\rangle_{\mathrm{BG}} = \alpha|\alpha, k\rangle_{\mathrm{BG}};$$

it can be expressed as [14]

$$|\alpha, k\rangle_{\mathrm{BG}} = \sqrt{\frac{|\alpha|^{2k-1}}{I_{2k-1}(2|\alpha|)}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!\Gamma(2k+n)}} |n, k\rangle, \tag{17}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind. For the case $k = \frac{1}{2}$, which we consider here, $\left|\alpha, \frac{1}{2}\right\rangle_{\mathrm{BG}}$ describes a nonlinear coherent state which may appear as a stationary state of the centre-of-mass motion of trapped ions [46].

The elements of the density operator in equation (11) are, initially in $(\kappa_e|e\rangle + \kappa_g|g\rangle)|\alpha, k\rangle_P$ or $(\kappa_e|e\rangle + \kappa_g|g\rangle)|\alpha, k\rangle_{\mathrm{BG}}$, or any $SU(1,1)$ coherent state as; $\hat{\rho}_{ee}(0) = |\kappa_e|^2|\alpha, k\rangle\langle\alpha, k|$, $\hat{\rho}_{gg}(0) = |\kappa_g|^2|\alpha, k\rangle\langle\alpha, k|$, $\hat{\rho}_{eg}(0) = \kappa_e\kappa_g^*|\alpha, k\rangle\langle\alpha, k| = (\hat{\rho}_{ge}(0))^\dagger$.

In order to obtain $\hat{\rho}(t)$, we find the matrix elements in question and substitute them into the expression

$$\hat{\rho}(t) = \sum_{m,n=0}^{\infty} \big[|\kappa_e|^2\big(p_{m,n} + \chi^{+}_{m,n}\big)\,\mathrm{e}^{-z^{++}_{m,n}(t)}|e\rangle\langle e| + \kappa_e\kappa_g^*(p_{m,n} + y_{m,n})\,\mathrm{e}^{-z^{+-}_{m,n}(t)}|e\rangle\langle g| + \kappa_g\kappa_e^*(p_{m,n} + y^*_{m,n})\,\mathrm{e}^{-z^{-+}_{m,n}(t)}|g\rangle\langle e| + |\kappa_g|^2\big(p_{m,n} + \chi^{*+}_{m,n}\big)\,\mathrm{e}^{-z^{--}_{m,n}(t)}|g\rangle\langle g|\big]|m,n\rangle\langle m,n|, \tag{18}$$

where

$$\begin{aligned} &z^{\pm\pm}_{m,n}(t) = \mu(t) \pm \mathrm{i}\eta^{\pm\pm}(t), \qquad z^{\pm\mp}_{m,n}(t) = \mu(t) \pm \mathrm{i}\eta^{\pm\mp}(t),\\ &\mu(t) = \gamma t[m^2 + n^2 + (m+n)(2k-1)],\\ &\eta^{\pm\mp}(t) = \lambda t[m(m+2k\pm 1) + n(n+2k\mp 1) + 2k],\\ &\eta^{\pm\pm}(t) = \lambda t[(m-n)(m+n+2k\pm 1)], \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\chi^{\pm}_{m,n} = \sum_{r=1}^{\infty}\frac{(2\gamma)^r}{r!}p_{m+r,n+r}\prod_{s=0}^{r-1}\big(A^{\pm}_{m+s+1,n+s+1} + iB^{\pm}_{m+s+1,n+s+1}\big),\\ &A^{\pm}_{m,n} = \beta_{m,n}\frac{x(1-\mathrm{e}^{-x}\cos y^{\pm}) + y^{\pm}\,\mathrm{e}^{-x}\sin y^{\pm}}{x^2 + y^{\pm 2}},\\ &B^{\pm}_{m,n} = \beta_{m,n}\frac{x\,\mathrm{e}^{-x}\sin y^{\pm} - y^{\pm}(1-\mathrm{e}^{-x}\cos y^{\pm})}{x^2 + y^{\pm 2}},\\ &x = 2\gamma t(m+n+2k-2), \qquad y^{+} = 2\lambda t(m-n), \qquad y^{-} = 2\lambda t(m+n+2k-1), \end{aligned} \tag{20}$$

with $\beta_{m,n} = \sqrt{m(m+2k-1)}\sqrt{n(n+2k-1)}$ and $p_{m,n}$ is the photon-number distribution of any $SU(1,1)$ coherent state ($|\alpha,k\rangle_P$ or $|\alpha,k\rangle_{\mathrm{BG}}$), with the Bargmann number $k = \frac{1}{2}$. The function $\mu(t)$, the real part of the exponential $\mathrm{e}^{-z^{++}_{m,n}(t)}$, in equation (19) embodies the effect of the reservoir because it vanishes as $\gamma \to 0$. We note that the exponential is present in all the coefficients of the density matrix; hence, the coherence properties of the field, atom and atom–field system states are affected by the damping. By these solutions, we investigate the dynamical properties of the dissipative nonlinear JC model.

3.3. Various measures of entanglement

There are different measures that have been used to quantify entanglement and correlation. The interaction between the system and its environment makes the system evolve from a pure state to a mixed state and leads to the decay of off-diagonal elements of the density operator, which means decoherence. The mixedness of the mixed state can be measured in terms of the linear entropy [37] or the total entropy S [22, 47]. The latter has a natural significance stemming from its connections with statistical physics and information theory [48]. An advantage of using the linear entropy instead of the total entropy is in the simplification of calculations, without changing the qualitative physical conclusions. In our work here, we use the total entropy S of the quantum-mechanical system described by the density operator $\hat{\rho}$, which quantifies the entanglement between the field–atom system and the environment. It is defined as

$$S = -\operatorname{Tr}\{\hat{\rho}\ln\hat{\rho}\}. \tag{21}$$

This entropy is computed by using numerical computations. There are other measures to quantify the temporal evolution of the entanglement (purity) between the states of the field or the atom. The purity of the state of the field is investigated by using the entropy of the marginal density matrix of the field $S_F = -\mathrm{Tr}\{\hat{\rho}_F \ln \hat{\rho}_F\} = -\sum_{i=1}^{\infty} \lambda_i^F \ln\left(\lambda_i^F\right)$. Here the eigenvalues λ_i^F of the reduced field density matrix $\hat{\rho}_F(t)$ are computed by using numerical calculations. The reduced atom density operator $\hat{\rho}_A(t) = \mathrm{Tr}_A\{\hat{\rho}(t)\}$ is given by $\hat{\rho}_A(t) = \mathrm{Tr}_F\{\hat{\rho}(t)\}$, and then the eigenvalues $\lambda_{1,2}^A$ for $\hat{\rho}_A(t)$ are

$$\lambda_{1,2}^A = \frac{1}{2} \pm \frac{1}{2}\sqrt{\left[\sum_i^{\infty}\left(\rho_{ii}^{ee} - \rho_{ii}^{gg}\right)\right]^2 + 4\left|\sum_i^{\infty}\rho_{ii}^{eg}\right|^2}. \tag{22}$$

So, the purity of the state of the atom is quantified by the atomic entropy, $S_A = -\lambda_1^A \ln\left(\lambda_1^A\right) - \lambda_2^A \ln\left(\lambda_2^A\right)$. If $S_A(S_F) = 0$, then the states are separable states.

To quantify the amount of entanglement (quantum correlation) of the final state (18), we use the concept of the negativity [49] defined by

$$E_N = \frac{\|\rho^T\| - 1}{2}, \tag{23}$$

where ρ^T is the matrix obtained by partially transposing the density matrix ρ, and $\|\rho^T\|$ is the trace class norm of the operator ρ^T. The trace class norm of any trace class operator $\hat{M}$ is defined by $\|\hat{M}\| = \mathrm{Tr}\sqrt{\hat{M}^\dagger \hat{M}}$ [50], which reduces to the sum of the absolute value of the eigenvalues of $\hat{M}$, when $\hat{M}$ is Hermitian. Therefore,

$$\|\rho^T\| = \sum_k \lambda_k - 2\sum_k \lambda_k^N = 1 - 2\sum_k \lambda_k^N, \tag{24}$$

where λ_k and λ_k^N are, respectively, the eigenvalues and the negative eigenvalues of ρ^T and we used also the fact that $\mathrm{Tr}(\rho^T) = \mathrm{Tr}(\rho) = 1$. When $E_N = 0$, the final state (18) is separable. Vidal and Werner [49] proved that the negativity E_N is an entanglement monotone and therefore it is a good measure of entanglement.

We shall consider another entropy functional to probe the amount of total correlations by using the mutual entropy or the entropy difference, $E_D = \frac{1}{2}(S_A + S_F - S)$ [51]. The mutual entropy is non-negative, and is zero if and only if the marginal states are not correlated. It is a measure of both quantum and classical correlation (total correlations) residing in the composite system.

The entanglement between the atom and the field, as well as the decoherence induced by the cavity of the dispersive IDJC model, was studied in [37]. It has been shown that, in the dispersive IDJC model, only the coherence of the atom is influenced by the cavity, though the atom does not couple to the cavity directly. The coherence of the field remains unchanged by the environment. However, in the nonlinear JC model considered here, the field is also affected by the cavity and its coherence will be lost due to the nonlinear dissipation.

In figures 1(*a*) and (*c*) and 2(*a*) and (*c*), we display the influence of the nonlinear dissipation on the temporal evolution of the total entropy S, the field entropy S_F and the entropy of the atom S_A as functions of the scaled time λt for two different $SU(1,1)$ coherent states. The presence of the local maxima and minima in the temporal evolution of S_A and S_F is due to the field and the atom loss and gain of their coherence. Because of the reservoir effect, we find that oscillations and amplitudes of both the field and the atom entropies damp and change with time. If we follow the curve of atomic entropy, we find that for any (however small) decay parameter γ, the atomic subsystem goes, asymptotically, to a completely mixed state with maximal possible entropy. Whereas the entropy of the field mode, with nonlinear dissipation, splits up from the atomic entropy and the amplitudes of the local maxima and minima decrease with time. The function $\mu(t)$ on the exponential $\mathrm{e}^{-z_{m,n}(t)}$ in equation (19) controls the coherence loss of the field. The real part of the exponential is always nonpositive; hence, it decreases

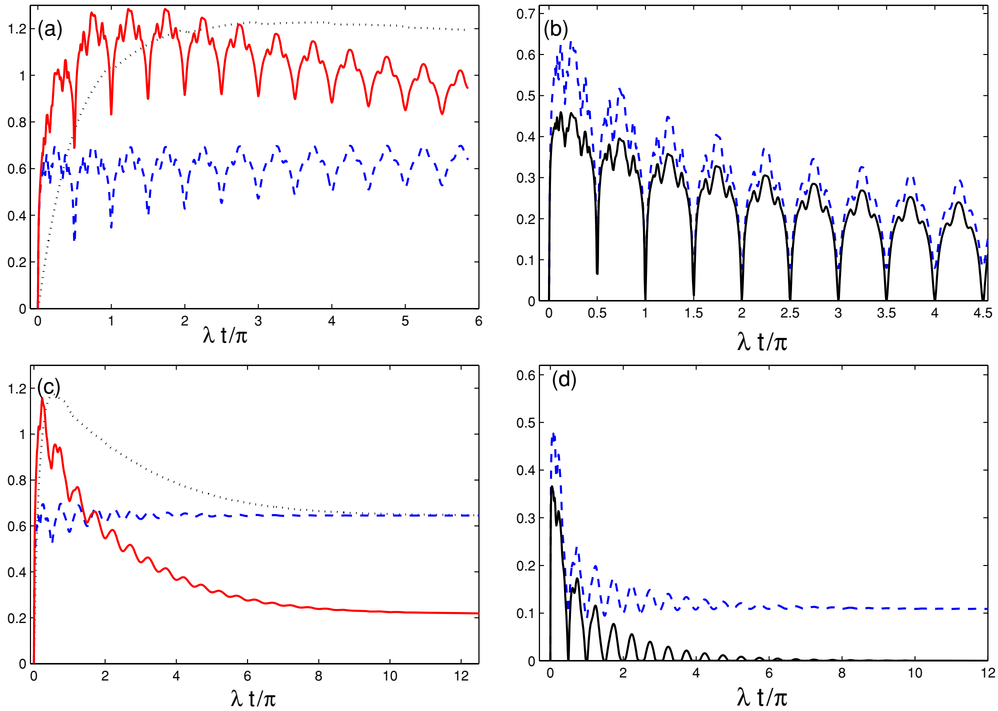

Figure 1. Time evolutions of S (dot curve), S_A (dash curve) and S_F (solid curve) as shown in (a) and (c) and the time evolution of $E_N(t)$ (solid curve) in comparison to $E_D(t)$ (dashed curve) as shown in (b) and (d), for different values of $\frac{\gamma}{\lambda} = 0.01$ (a, b), $\frac{\gamma}{\lambda} = 0.1$ (c, d) with $|\alpha|^2 = 0.8$ for the Perelomov $SU(1, 1)$ coherent state, with $\kappa_e = \kappa_g = \frac{1}{\sqrt{2}}$.

with time. With larger γ, the field finally goes into a mixed state and its coherence is lost completely. One finds that for larger γ, the amplitudes are quickly suppressed; S, S_F and S_A reach their minimum, maximum values and asymptotic values rapidly (see figures 1(c) and 2(c)).

Since the partial entropies for the atom and the field are no longer equal, they cannot be used as measures for entanglement between the atom and the field. Therefore, we use the negativity and the mutual entropy as anagrammatic measures to quantify the entanglement between the atom and the field. Figures 1(b) and (d) and 2(b) and (d) show the time development of the negativity and the mutual entropy for $|\alpha, k\rangle_P$ and $|\alpha, k\rangle_{\mathrm{BG}}$. Comparing E_N and E_D in these figures we find complete agreement in the trend between the two measures as long as we deal with a pure state. The main difference between the two measures is only the difference between the amplitudes of oscillations. In figures 1 and 2, we display the entanglement for both $|\alpha, k\rangle_P$ and $|\alpha, k\rangle_{\mathrm{BG}}$ with the same mean photon number $|\alpha|^2 = 0.8$. From these figures we find that the measures of entanglement with $|\alpha, k\rangle_P$ are more affected than those for $|\alpha, k\rangle_{\mathrm{BG}}$. For example, with $|\alpha, k\rangle_P$ and at weak damping $\gamma = 0.01$ the field entropy splits from the atomic entropy faster than its counterpart in the case of $|\alpha, k\rangle_{\mathrm{BG}}$. As is observed from figures 1 and 2, we find that the local maxima and minima occur at the same periods. Therefore, the used measures give the same period of the entanglement. Finally, from these figures, the values of the negativity disappear faster than its counterpart of the mutual entropy.

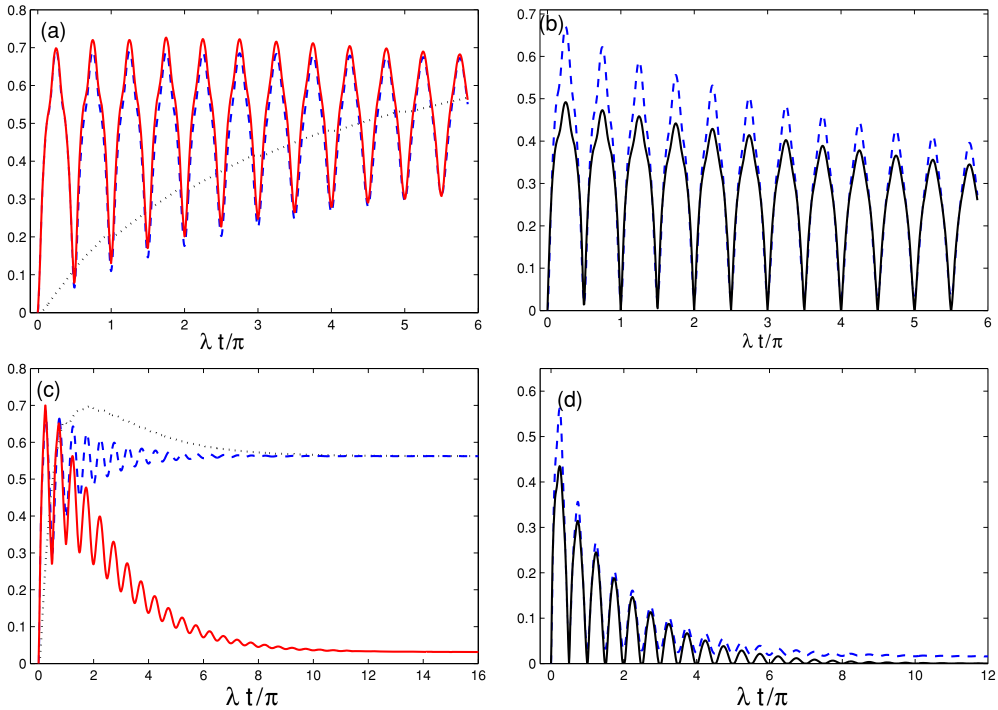

Figure 2. The same as in figure 1 but for the Barut–Girardello $SU(1, 1)$ coherent state.

This implies that the entanglement of the atom and the above field states is sensitive to the nonlinear dissipation, and to the type of the state.

Figures 1, 2(*b*) and (*d*) show that, for a long period of time, one measure (the negativity) may drop to zero, while the other (the mutual entropy) remains finite. The mutual information is non-zero does not *a priori* mean that there is (quantum) entanglement in the system, since it can be entirely due to classical correlations. This is because the mutual entropy is approximately equal to the classical upper bound min $[S_A, S_F]$, i.e. the bound that becomes saturated when the two subsystems A and F are classically maximally correlated. Since only the range between classical and quantum upper bounds corresponds to pure quantum entanglement [52], it appears that for large values of γ, the mutual entropy evaluation of the system shows that the system is more classically correlated than quantum correlated.

It is worth noting that the output correlation between the system and the environment S with *nonlinear dissipation* grows with time to its maximum value (which oversteps the atomic entropy S_A) and decreases to a steady curve which tends to the steady curve of the atomic entropy. This means that the system environment loses purity with nonlinear dissipation and falls into a statistically mixed state. These results differ from their counterpart with *linear dissipation* [37], in which S increases monotonically and it reaches directly to (without overstepping S_A) the steady curve of the atomic entropy, which means that the system environment falls into a statistically mixed state without partial gain of purity.

4. Conclusions

The master equation in the dispersive limit and the nonlinear dissipation for an initial field obeying the $SU(1, 1)$ Lie algebra is solved. Considering the field to be initially in $SU(1, 1)$

coherent states, it is found that in the presence of nonlinear quantum dissipation, the amplitude of the entanglement between the field and the atom decreases with time, and the entanglement between the cavity mode-atom system and the environment grows with time. From comparing E_N and E_D, a complete agreement in the trend between the negativity and the quantum mutual entropy is noted. But for long times, one measure (the negativity) may drop to zero, while the other (the mutual entropy) remains finite. It is worth noting that one of the pioneering experimental works on the JC-like dynamics in the context of trapped ions was reported in [53]. They observed the Rabi oscillations among two hyperfine levels of a $^9Be^+$ ion by measuring the probability of finding the ion in its lower electronic level. Results of this paper may be relevant to such experiments. In order to gain insight into processes of entanglement, dissipation and the relation between them, we need to evaluate the time evolution of any initial states for an atom and the field, where the solution which is introduced by the $SU(1, 1)$ algebra will allow this.

Acknowledgments

The authors wish to thank the referees for their valuable comments that resulted in improvements of the paper in many aspects.

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